Solution to Math4230 Tutorial 11

1. Consider the convex programming problem

$$\min\ f(x)$$
$$st.\ x \in X,\ \ g(x) \le 0,$$

assume that the set X is described by equality and inequality constraints as

$$X = \{x | h_i(x) = 0, i = 1, \cdots, \bar{m}, g_j(x) \le 0, j = r+1, \cdots, \bar{r}\}.$$

Then the problem can alternatively be described without an abstract set constraint in terms of all of the constraint functions

$$h_i(x) = 0, i = 1, \cdots, \bar{m}, \quad g_j(x) \le 0, j = 1, \cdots, \bar{r}.$$

We call this the extended representation of primal problem. Show if there is no duality gap and there exists a dual optimal solution for the extended representation, the same is true for the original problem

Solution.
Assume that there exists a dual optimal solution in the extended representation. Thus there exist nonnegative scalars $\lambda_1^*, \ldots, \lambda_m^*, \lambda_{m+1}^*, \ldots, \lambda_{\bar{m}}^*$ and $\mu_1^*, \ldots, \mu_r^*, \mu_{r+1}^*, \ldots, \mu_{\bar{r}}^*$ such that

$$f^* = \inf_{x \in R^n} \left\{ f(x) + \sum_{i=1}^{\bar{m}} \lambda_i^* h_i(x) + \sum_{j=1}^{\bar{r}} \mu_j^* g_j(x) \right\},$$

from which we have

$$f^* \le f(x) + \sum_{i=1}^{\bar{m}} \lambda_i^* h_i(x) + \sum_{j=1}^{\bar{r}} \mu_j^* g_j(x), \qquad \forall\ x \in R^n.$$

For any $x \in X$, we have $h_i(x) = 0$ for all $i = 1, \ldots, \bar{m}$, and $g_j(x) \le 0$ for all $j = r+1, \ldots, \bar{r}$, so that $\mu_j^* g_j(x) \le 0$ for all $j = r+1, \ldots, \bar{r}$. Therefore, it follows from the preceding relation that

$$f^* \le f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x), \qquad \forall\ x \in X.$$

Taking the infimum over all $x \in X$, it follows that

$$\begin{aligned} f^* &\le \inf_{x \in X} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} \\ &\le \inf_{x \in X, g_j(x) \le 0,\ j=1,\ldots,r} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} \\ &\le \inf_{\substack{x \in X,\ h_i(x)=0,\ i=1,\ldots,m \\ g_j(x) \le 0,\ j=1,\ldots,r}} f(x) \\ &= f^*. \end{aligned}$$

Hence, equality holds throughout above, showing that the scalars $\lambda_1^*, \ldots, \lambda_m^*$, $\mu_1^*, \ldots, \mu_r^*$ constitute a dual optimal solution for the original representation.

2. A simple example: consider the optimization problem

$$\min\ x^2 + 1$$
$$st.\ (x-2)(x-4) \le 0,$$

with variable $x \in \mathbb{R}$.

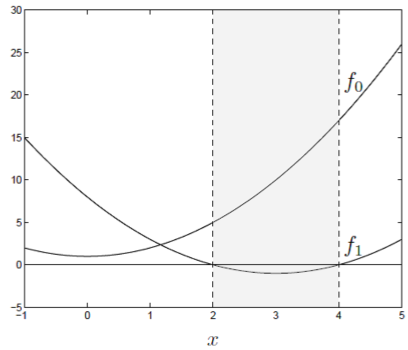
(a) Analysis of primal problem. Give the feasible set, the optimal value, and the optimal solution.

(b) Lagrangian and dual function. Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x,\lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \ge \inf_x L(x,\lambda)$) for $\lambda \ge 0$). Derive and sketch the Lagrange dual function g.

(c) Lagrange dual problem. State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^*. Does strong duality hold?

Solution:

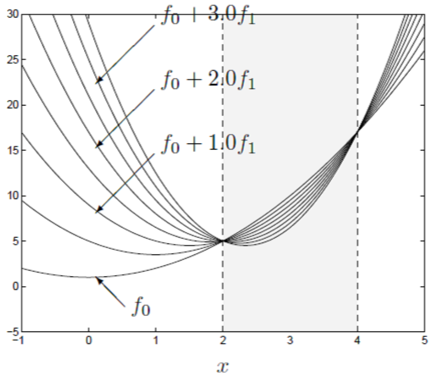
(a) The feasible set is the interval $[2,4]$. The (unique) optimal point is $x^* = 2$ and the optimal value is $p^* = 5$. The plot shows f_0 and f_1.



(b) The Lagrangian is

$$L(x,\lambda) = (1+\lambda)x^2 - 6\lambda x + (1+8\lambda).$$

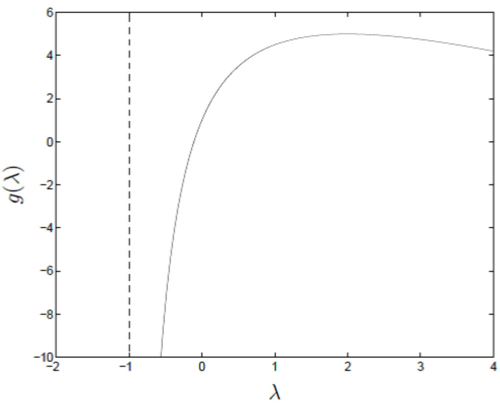
The plot shows the Lagrangian $L(x,\lambda) = f_0 + \lambda f_1$ as a function of x for different values of $\lambda \ge 0$. Note that the minimum value of $L(x,\lambda)$ over x (*i.e.*, $g(\lambda)$) is always less than $p^\star$. It increases as λ varies from 0 toward 2, reaches its maximum at $\lambda = 2$, and then decreases again as λ increases above 2. We have equality $p^\star = g(\lambda)$ for $\lambda = 2$.



For $\lambda > -1$, the Lagrangian reaches its minimum at $\tilde{x} = 3\lambda/(1+\lambda)$. For $\lambda \leq -1$ it is unbounded below. Thus

$$g(\lambda) = \begin{cases} -9\lambda^2/(1+\lambda) + 1 + 8\lambda & \lambda > -1 \\ -\infty & \lambda \leq -1 \end{cases}$$

which is plotted below.



We can verify that the dual function is concave, that its value is equal to $p^\star = 5$ for $\lambda = 2$, and less than $p^\star$ for other values of λ.

(c) The Lagrange dual problem is

$$\begin{array}{ll} \text{maximize} & -9\lambda^2/(1+\lambda) + 1 + 8\lambda \\ \text{subject to} & \lambda \geq 0. \end{array}$$

The dual optimum occurs at $\lambda = 2$, with $d^\star = 5$. So for this example we can directly observe that strong duality holds (as it must — Slater's constraint qualification is satisfied).

3. Consider the convex programming problem

$$\begin{aligned} &\min\ f(x) \\ &\ st.\ x \in X,\ \ g(x) \leq u_j, j = 1, \cdots, r, \end{aligned}$$

where $u = (u_1, \cdots, u_r)$ is a vector parameterizing the right-hand side of the constraints. Given two distinct values $\bar{u}$ and $\tilde{u}$ of u, let $\bar{f}$ and $\tilde{f}$ be the corresponding optimal values, and assume that $\bar{f}$ and $\tilde{f}$ are finite. Assume further that $\bar{\mu}$ and $\tilde{\mu}$ are corresponding dual optimal solutions and that there is no duality gap. Show that

$$\tilde{\mu}'(\tilde{u} - \bar{u}) \leq \bar{f} - \tilde{f} \leq \bar{\mu}'(\tilde{u} - \bar{u}).$$

Solution:

We have

$$\bar{f} = \inf_{x \in X} \{f(x) + \bar{\mu}'(g(x) - \bar{u})\},$$

$$f = \inf_{x \in X} \{f(x) + \mu'(g(x) - u)\}.$$

Let $\bar{q}(\mu)$ denote the dual function of the problem corresponding to $\bar{u}$:

$$\bar{q}(\mu) = \inf_{x \in X} \{f(x) + \mu'(g(x) - \bar{u})\}.$$

We have

$$\begin{aligned}
\bar{f} - f &= \inf_{x \in X} \{f(x) + \bar{\mu}'(g(x) - \bar{u})\} - \inf_{x \in X} \{f(x) + \mu'(g(x) - u)\} \\
&= \inf_{x \in X} \{f(x) + \bar{\mu}'(g(x) - \bar{u})\} - \inf_{x \in X} \{f(x) + \mu'(g(x) - \bar{u})\} + \mu'(u - \bar{u}) \\
&= \bar{q}(\bar{\mu}) - \bar{q}(\mu) + \mu'(u - \bar{u}) \\
&\geq \mu'(u - \bar{u}),
\end{aligned}$$

where the last inequality holds because $\bar{\mu}$ maximizes $\bar{q}$.

This proves the left-hand side of the desired inequality. Interchanging the roles of $\bar{f}$, $\bar{u}$, $\bar{\mu}$, and f, u, μ, shows the desired right-hand side.